Calcus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 27

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Textbook Sections: 11.11, 11.12

Todays Topic: Manipulating Taylor Series, and Applications

In the first portion of todays lecture, lets do a few more examples to get warmed up.

Recall our general goal: Understand when a function $f(x)$ is equal to its Taylor series on an interval $|x-a|<R$. In other words, we want to know when its true that

$$f(x) = \lim_{N\to\infty} T_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all $|x-a|<R$.

To help assess this situation, we can use Taylor series remainder estimates:

Theorem (Taylor's Inequality)

Suppose $f^{(N+1)}(x)$ is continuous on $|x-a|<R$, and that $|f^{(N+1)}(x)| \le M < \infty$ there.

Then $$|R_N(x)| = \left|f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n\right| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Last class, we saw how to use this to prove that

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{on } (-\infty,\infty).$$

lets do a few more examples:

Examples:

6) e^x is analytic. (from last time)

We need to show that $R_N(x) = e^x - \sum_{n=0}^{N} \frac{1}{n!}x^n$ has $|R_N(x)| \to 0$ for all $x \in \mathbb{R}$ as $N \to \infty$.

(i) Notice that $\frac{d^n}{dx^n} e^x = e^x$. So, if $|x| \le R$ (for any fixed $R > 0$), $\left|\frac{d^n}{dx^n} e^x\right| = |e^x| = e^x \le e^R$ for any $n \ge 1$.

(ii) By Taylor's Inequality, for any $|x| \le R$

$$0 \le |R_N(x)| \le \frac{e^R}{(N+1)!}|x|^{N+1} = \frac{e^R \cdot R^{N+1}}{(N+1)!} \to 0 \quad \text{as } N \to \infty.$$

(iii) Thus, since $R > 0$ could have been any fixed positive number, we see that $|R_N(x)| \to 0$ at all $x \in \mathbb{R}$.

(iv) Thus, $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ for all $x \in \mathbb{R}$.

9) Find the Taylor series of e^x centered at $a = 1$:

(i) As above, $\frac{d^n}{dx^n} e^x\Big|_{x=1} = e^x\Big|_{x=1} = e$, so we have

$$\sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$$

for the Taylor series of e^x centered at $a = 1$.

(ii) Using the same argument as above, we can show that in this case we also have $|R_N(x)| \to 0$ as $N \to \infty$ for all $x \in \mathbb{R}$. Thus, we have another series representation of

$$e^x = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$$

(iii) From the point of view of numerical approximation, its better to use a series with basepoint close to the values of x we are interested in, since the remainder terms are smaller!

Here's a new example

10) Fix $k \in \mathbb{R}$. Find the Taylor series of $f(x) = (1+x)^k$ at $a=0$.

(i) Let's just start taking derivatives and evaluating at $x=0$:

$$f^{(0)}(x) = f(x) \rightsquigarrow f^{(0)}(0) = 1$$

$$f'(x) = k(1+x)^{k-1} \rightsquigarrow f'(0) = k$$

$$f''(x) = k(k-1)(1+x)^{k-2} \rightsquigarrow f''(0) = k(k-1)$$

$$\vdots$$

$$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n} \rightsquigarrow f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

(ii) The Taylor series of $f(x)$ is thus

$$f(x) = (1+x)^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

Def/Notation: $\binom{k}{n} := \frac{k(k-1)\cdots(k-n+1)}{n!}$ (nb. $\binom{k}{0} := 1$)

Case 1: If $k \in \mathbb{Z}^{\geq 0}$, eventually (for n big enough), all these coefficients are 0 since they contain the factor $(k-(k+1)+1) = 0$. Indeed, in this case $(1+x)^k$ is simply a polynomial, and the Taylor series of $(1+x)^k$ is nothing other than the expansion of this product!

In this case, clearly $f(x)$ is <u>equal</u> to its Taylor series.

Case 2: If $k \notin \mathbb{Z}^{\geq 0}$, all of these coefficients are nonzero, and the Taylor series of $f(x)$ has ∞'ly many terms.

In this case, it's not as clear that $f(x)$ is equal to its Taylor series on any interval around $x=0$.

(iii) **Exercise 1**: Use the ratio test to show that the

binomial series $\sum_{n=0}^{\infty} \binom{k}{n} x^n$ converges on $|x|<1$ and

- @ ± 1 if $k \geq 0$
- @ 1 if $-1 < k \leq 0$

Skip in lecture

Exercise 2: (Challenge) Of course, this still doesn't tell us if

$$f(x) = (1+x)^k \stackrel{?}{=} \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

at these points. To check that, we would need to verify that

$$|R_N(x)| \to 0 \quad \text{as } N \to \infty$$

for these choices of x. This is tricky using Taylor's formula, but here's a sneaky approach:

① On $|x|<1$, $g(x) := \sum_{n=0}^{\infty} \binom{k}{n} x^n$ defines a smooth function (since we checked that with the ratio test!).

$f(x) = (1+x)^k$ is also smooth there, so it satisfies:

$$f(x) = (1+x)^k \Rightarrow f'(x) = k(1+x)^{k-1} = \frac{k(1+x)^k}{1+x} = \frac{k f(x)}{1+x}$$

② Let's see if $g(x)$ satisfies the same ODE: we can compute

$$\begin{aligned}
(1+x)\cdot g'(x) &= (1+x)\sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1} = (1+x)\sum_{n=0}^{\infty} \binom{k}{n+1}(n+1)x^n \\
&= \sum_{n=0}^{\infty} \frac{k\cdot(k-1)\cdots(k-n)}{(n+1)!}(n+1)!\,(x^n + x^{n+1}) \\
&= \sum_{n=0}^{\infty} k\frac{(k-1)\cdots((k-1)-n+1)}{n!}x^n + \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n)}{n!}x^{n+1} \\
&= \sum_{n=0}^{\infty} k\binom{k-1}{n}x^n + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{(n-1)!}x^n \\
&= k + \sum_{n=1}^{\infty}\left\{k\binom{k-1}{n} + n\binom{k}{n}\right\}x^n \\
&= k\left(1 + \sum_{n=1}^{\infty}\left\{\binom{k-1}{n} + \frac{n}{k}\binom{k}{n}\right\}x^n\right) \\
&= k\left(1 + \sum_{n=1}^{\infty}\left\{\binom{k-1}{n} + \binom{k-1}{n-1}\right\}x^n\right) \\
&= k\left(1 + \sum_{n=1}^{\infty}\binom{k}{n}x^n\right) \\
&= k\,g(x)
\end{aligned}$$

Binomial coefficient identities from combinatorics

(3) Because $f(x)$ and $g(x)$ both agree at $x=0$, and satisfy the same ODE, they must be equal! Indeed, if we set $h(x) = \frac{g(x)}{(1+x)^k}$, then

$$h'(x) = \frac{(1+x)^k g'(x) - g(x)\cdot\left[(1+x)^k\right]'}{(1+x)^{2k}} = 0$$

$\oplus$ $h(0) = 1$

$$\rightsquigarrow\quad 1 \equiv h(x) = \frac{g(x)}{(1+x)^k}.$$

Application: 11) Find the Taylor series of $f(x) = \frac{1}{\sqrt{4-x}}$:

Solution: Notice that $f(x) = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$. Thus, using the binomial series with $k = -\frac{1}{2}$,

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(-\frac{x}{4}\right)^n.$$

This converges when $-1 < -\frac{x}{4} \leq 1$ (since $-1 < k \leq 0$), i.e. when $-4 \leq x < 4$.

Lastly for today, we mention that one can do multiplication and long division with Taylor series, just like for polynomials:

Theorem: Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} d_n(x-a)^n$ are both convergent power series on $|x-a| < R$

(1) $f(x)\cdot g(x) = \left(\sum_0^\infty c_n(x-a)^n\right)\left(\sum_0^\infty d_n(x-a)^n\right)$

$$= c_0d_0 + (c_1d_0 + c_0d_1)(x-a) + (c_2d_0 + c_1d_1 + c_0d_2)(x-a)^2 + \cdots$$

converges on $|x-a| < R$.

(2) If $g(a) = d_0 \neq 0$, then

$$\frac{f(x)}{g(x)} = \frac{\sum_0^\infty c_n(x-a)^n}{\sum_0^\infty d_n(x-a)^n} = \text{--- long division converges for } |x-a| < R.$$

Examples / Exercises:

12) Compute $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

Insert the Taylor series of e^x and divide:

$$\frac{e^x - 1 - x}{x^2} = \frac{\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x}{x^2} = \sum_{n=2}^{\infty} \frac{1}{n!} x^{n-2} = \frac{1}{2} + \sum_{n=3}^{\infty} \frac{x^{n-2}}{n!}$$

Thus take $\lim_{x \to 0} \left(\frac{1}{2} + \sum_{n=3}^{\infty} \frac{x^{n-2}}{n!} \right) = \frac{1}{2}$.

13) Compute the Taylor series of $\sin(x)\cos(x)$:

$$\sin(x)\cos(x) = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right)$$

$$= \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right)$$

$$= x - \left(\frac{1}{3!} + \frac{1}{2!} \right) x^3 + \left(\frac{1}{2!}\frac{1}{3!} + \frac{1}{5!} + \frac{1}{4!} \right) x^5 + \cdots$$

14) Compute the Taylor series of $\tan(x)$:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \right)}{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right)} = x + \frac{1}{3}x^3 + \frac{2}{15}x^3 + \cdots$$

Since:

$$\begin{array}{r} x + \frac{1}{3}x^3 + \frac{2}{15}x^3 + \cdots \\ \hline 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \;\Big)\; x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots \\ x - \frac{1}{2}x^3 + \frac{x^5}{24} \\ \hline \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\ \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\ \hline \frac{2}{15}x^5 + \cdots \end{array}$$